Chapter 10

Vectors

Types of Vectors

A **vector** has direction and magnitude both but scalar has only magnitude.

Magnitude of a vector a is denoted by |a| or a. It is non-negative scalar.

Equality of Vectors

Two vectors a and b are said to be equal written as a = b, if they have (i) same length (ii) the same or parallel support and (iii) the same sense.

Types of Vectors

(i) Zero or Null Vector A vector whose initial and terminal points are coincident is called zero or null vector. It is denoted by 0.

(ii) Unit Vector A vector whose magnitude is unity is called a unit vector which is denoted by $\hat{n}$

(iii) Free Vectors If the initial point of a vector is not specified, then it is said to be a free vector.

(iv) Negative of a Vector A vector having the same magnitude as that of a given vector a and the direction opposite to that of a is called the negative of a and it is denoted by —a.

(v) Like and Unlike Vectors Vectors are said to be like when they have the same direction and unlike when they have opposite direction.

(vi) Collinear or Parallel Vectors Vectors having the same or parallel supports are called collinear vectors.

(vii) Coinitial Vectors Vectors having same initial point are called coinitial vectors.

(viii) Coterminous Vectors Vectors having the same terminal point are called coterminous vectors.

(ix) Localized Vectors A vector which is drawn parallel to a given vector through a specified point in space is called localized vector.

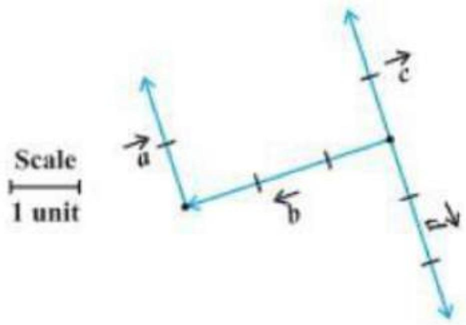
(x) Coplanar Vectors A system of vectors is said to be coplanar, if their supports are parallel to the same plane. Otherwise they are called non-coplanar vectors.

(xi) Reciprocal of a Vector A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector is known as the reciprocal of a.

i.e., if $|a| = a$, then $|a^{-1}| = 1 / a$.

Example 1

In the figure given below, identify Collinear, Equal and Coinitial vectors:



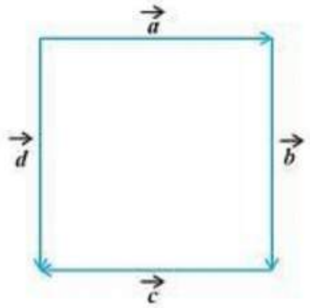
Solution: By definition, we know that

- Collinear vectors are two or more vectors parallel to the same line irrespective of their magnitudes and direction. Hence, in the given figure, the following vectors are collinear: $\vec{a}$, $\vec{c}$, and $\vec{d}$.
- Equal vectors have the same magnitudes and direction regardless of their initial points. Hence, in the given figure, the following vectors are equal: $\vec{a}$ *and* $\vec{c}$
- Coinitial vectors are two or more vectors having the same initial point. Hence, in the given figure, the following vectors are coinitial: $\vec{b}$, $\vec{c}$, and $\vec{d}$.

More Solved Examples

Question: In the given figure, identify the following vectors

1. Coinitial
2. Equal
3. Collinear but not equal



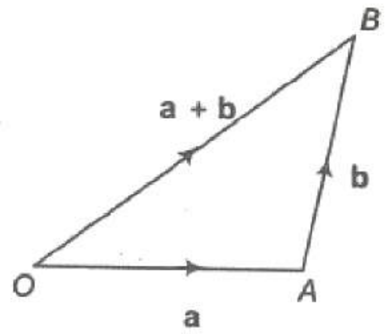
Solution:

- Coinitial vectors have the same initial point. In the figure given above, vectors $\vec{a}$ *and* $\vec{d}$ are the two vectors which have the same initial point P.
- Equal vectors have same magnitudes and direction. In the figure given above, vectors $\vec{b}$ *and* $\vec{d}$ are equal vectors.
- Collinear vectors are two or more vectors parallel to the same line. In the figure given above, vectors $\vec{a}$ *and* $\vec{c}$ are parallel and hence, collinear. Also, vectors $\vec{b}$ *and* $\vec{d}$ are parallel and hence, collinear. We know that vectors $\vec{b}$ *and* $\vec{d}$ are also equal. Hence, vectors $\vec{a}$ *and* $\vec{c}$ are collinear but not equal.

Addition and Multiplication of a Vector by a Scalar

Addition of Vectors

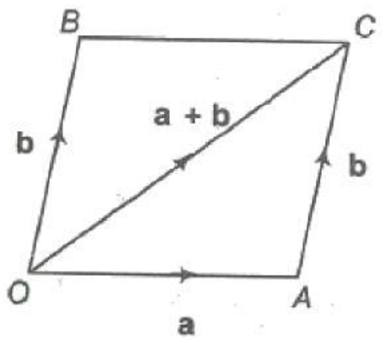
Let **a** and **b** be any two vectors. From the terminal point of a, vector b is drawn. Then, the vector from the initial point O of a to the terminal point B of b is called the sum of vectors a and b and is denoted by **a + b**. This is called the triangle law of addition of vectors.



Parallelogram Law

Let a and b be any two vectors. From the initial point of a, vector b is drawn and parallelogram OACB is completed with OA and OB as adjacent sides. The vector OC is defined as the sum of a and b. This is called the parallelogram law of addition of vectors.

The sum of two vectors is also called their resultant and the process of addition as composition.



Properties of Vector Addition

(i) $a + b = b + a$ (commutativity)

(ii) $a + (b + c) = (a + b) + c$ (associativity)

(iii) $a + 0 = a$ (additive identity)

(iv) $a + (-a) = 0$ (additive inverse)

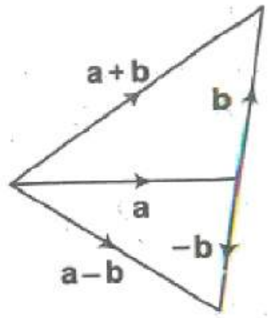
(v) $(k_1 + k_2)\, a = k_1 a + k_2 a$ (multiplication by scalars)

(vi) $k(a + b) = k a + k b$ (multiplication by scalars)

(vii) |a+ b| ≤ |a| + |b| and |a – b| ≥ |a| – |b|

Difference (Subtraction) of Vectors

If a and b be any two vectors, then their difference a – b is defined as a + (- b).



Multiplication of a Vector by a Scalar

Let a be a given vector and λ be a scalar. Then, the product of the vector a by the scalar λ is λ a and is called the multiplication of vector by the scalar.

Important Properties

(i) |λ a| = |λ| |a|

(ii) λ 0 = 0

(iii) m (-a) = – ma = – (m a)

(iv) (-m) (-a) = m a

(v) m (n a) = mn a = n(m a)

(vi) (m + n)a = m a+ n a

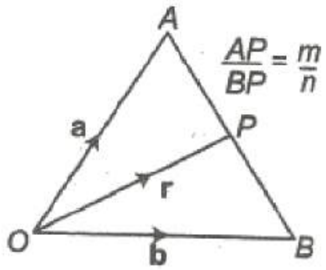
(vii) m (a+b) = m a + m b

Section Formula and Scalar Product of Two Vectors

Section Formula

Let A and B be two points with position vectors a and b, respectively and OP= r.

(i) Let P be a point dividing AB internally in the ratio m : n. Then, r = m b + n a / m + n



Also, (m + n) OP = m OB + n OA

(ii) The position vector of the mid-point of a and b is a + b / 2.

(iii) Let P be a point dividing AB externally in the ratio m : n. Then,

r = m b + n a / m + n

Position Vector of Different Centre of a Triangle

(i) If a, b, c be PV's of the vertices A, B, C of a ΔABC respectively, then the PV of the centroid G of the triangle is a + b + c / 3.

(ii) The PV of incentre of ΔABC is (BC)a + (CA)b + (AB)c / BC + CA + AB

(iii) The PV of orthocentre of ΔABC is

a(tan A) + b(tan B) + c(tan C) / tan A + tan B + tan C

Scalar Product of Two Vectors

If a and b are two non-zero vectors, then the scalar or dot product of a and b is denoted by a * b and is defined as a * b = |a| |b| cos θ, where θ is the angle between the two vectors and 0 < θ < π .

(i) The angle between two vectors a and b is defined as the smaller angle θ between them, when they are drawn with the same initial point.

Usually, we take 0 < θ < π.Angle between two like vectors is O and angle between two unlike vectors is π .

(ii) If either a or b is the null vector, then scalar product of the vector is zero.

(iii) If a and b are two unit vectors, then a * b = cos θ.

(iv) The scalar product is commutative

i.e., a * b= b * a

(v) If i , j and k are mutually perpendicular unit vectors i , j and k, then

i * i = j * j = k * k =1 and i * j = j * k = k * i = 0

(vi) The scalar product of vectors is distributive over vector addition.

(a) a * (b + c) = a * b + a * c (left distributive)

(b) (b + c) * a = b * a + c * a (right distributive)

Note Length of a vector as a scalar product

If a be any vector, then the scalar product

$a * a = |a|\ |a| \cos\theta \Rightarrow |a|^2 = a^2 \Rightarrow a = |a|$

Condition of perpendicularity a * b = 0 <=> a ⊥ b, a and b being non-zero vectors.

Important Points to be Remembered

(i) $(a + b) * (a - b) = |a|^22 - |b|^2$

(ii) $|a + b|^2 = |a|^22 + |b|^2 + 2\ (a * b)$

(iii) $|a - b|^2 = |a|^22 + |b|^2 - 2\ (a * b)$

(iv) $|a + b|^2 + |a - b|^2 = (|a|^22 + |b|^2)$ and $|a + b|^2 - |a - b|^2 = 4\ (a * b)$

or $a * b = 1 / 4\ [\ |a + b|^2 - |a - b|^2\ ]$

(v) If |a + b| = |a| + |b|, then a is parallel to b.

(vi) If |a + b| = |a| – |b|, then a is parallel to b.

(vii) $(a * b)^2 \leq |a|^22\ |b|^2$

(viii) If $a = a_1i + a_2j + a_3k$, then $|a|^2 = a * a = a_1^2 + a_2^2 + a_3^2$

Or

$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(ix) **Angle between Two Vectors** If θ is angle between two non-zero vectors, a, b, then we have

$a * b = |a|\,|b| \cos\theta$

$\cos\theta = a * b / |a|\,|b|$

If $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$

Then, the angle θ between a and b is given by

$\cos\theta = a * b / |a|\,|b| = a_1b_1 + a_2b_2 + a_3b_3 / \sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}$

(x) **Projection and Component of a Vector**

Projection of a on b = a * b / |a|

Projection of b on a = a * b / |a|

Vector component of a vector a on b

$$= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}^2|}\mathbf{b}$$

Similarly, the vector component of b on a = $((a * b) / |a^2|) * a$

(xi) **Work done by a Force**

The work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement.

∴ F * S = dot products of force and displacement.

Suppose F_1, F_1,..., F_n are n forces acted on a particle, then during the displacement S of the particle, the separate forces do quantities of work $F_1 * S$, $F_2 * S$, $F_n * S$.

Total workdone is $\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{S} = \sum_{i=1}^{n} \mathbf{S} \cdot \mathbf{F}_i = \mathbf{S} \cdot \mathbf{R}$

Here, system of forces were replaced by its resultant R.

Vector Triple Product

(b) Vector Triple Product

Consider next the cross product of $\vec{a}$ and $\vec{b} \times \vec{c}$, viz. $\vec{p} = \vec{a} \times (\vec{b} \times \vec{c})$.

This is a vector perpendicular to both a $\vec{a}$ and $\vec{b} \times \vec{c}$. But $\vec{b} \times \vec{c}$ is normal to the plane of $\vec{b}$ and $\vec{c}$, so that $\vec{p}$ must lie in this plane. It is therefore expressible in terms of $\vec{b}$ and $\vec{c}$, in the form $\vec{p} = \ell\vec{b} + m\vec{c}$. To find the actual expression for $\vec{p}$ consider unit vectors j^ and k^ the first parallel to $\vec{p}$ and the second perpendicular to it in the plane $\vec{b}$, $\vec{c}$. Then we may put $\vec{b} = b_1\hat{j}$, $\vec{c} = c_2\hat{j} + c_3\hat{k}$.

In terms of j^ and k^ and the other unit vector î of the right-handed system, the remaining vector $\vec{a}$ may be written $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then $\vec{b} \times \vec{c} = bc_3\hat{i}$, Then $\vec{b} \times \vec{c} = bc_3\hat{i}$, and the triple product

$$\vec{a} \times (\vec{b} \times \vec{c}) = a_3b_1c_3\hat{j} - a_2b_1c_3\hat{k} = (a_2c_2 + a_3c_3)\, b_1\hat{j} - a_2b_1(c_2\hat{j} + c_3\hat{k})$$

$$= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \quad ...(1)$$

This is the required expression for $\vec{p}$ in terms of $\vec{b}$ and $\vec{c}$,

Similarly the triple product $(\vec{b} \times \vec{c}) \times \vec{a} = -\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$. ...(2)

It will be noticed that the expansions (1) and (2) are both written down by the same rule. Each scalar product involves the factor outside the bracket; and the first is the scalar product of the extremes.
In a vector triple product the position of the brackets cannot be changed without altering the value of the product. For $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector expressible in terms of $\vec{a}$ and $\vec{b}$; $\vec{a} \times (\vec{b} \times \vec{c})$ is one expressible in terms of $\vec{b}$ and $\vec{c}$, The products in general therefore represent different vectors. If a vector r is resolved into two

others in the plane of $\vec{a}$ and $\vec{r}$, one parallel to and the other perpendicular to it, the former is $\frac{\vec{a}.\vec{r}}{\vec{a}^2}\vec{a}$, and therefore the latter $\frac{\vec{a}.\vec{r}}{\vec{a}^2} = \frac{(\vec{a}.\vec{a})\vec{r}-(\vec{a}.\vec{r})\vec{a}}{\vec{a}^2} = \frac{\vec{a}\times(\vec{r}\times\vec{a})}{\vec{a}^2}$

Geometrical Interpretation of $(\vec{a} \times \vec{b}) \times \vec{c}$

Consider the expression $(\vec{a} \times \vec{b}) \times \vec{c}$ which itself is a vector, since it is a cross product of two vectors $\vec{a}\ \&\ (\vec{b}\times\vec{c})$. Now $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector perpendicular to the plane containing $\vec{a}\ \&\ (\vec{b}\times\vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane $\vec{b}$ and $\vec{c}$, therefore $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector lies in the plane of $\vec{b}$ and $\vec{c}$, and perpendicular to a . Hence we can express $(\vec{a} \times \vec{b}) \times \vec{c}$ in terms of $\vec{b}$ and $\vec{c}$, i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

Note :

(i) $\vec{a} \times (\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(ii) $(\vec{a}\times\vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(iii) $(\vec{a}\times\vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b}\times\vec{c})$

Ex.24 Find a vector $\vec{v}$ which is coplanar with the vectors $\hat{i} + \hat{j} - 2\hat{k}\ \&\ \hat{i} - 2\hat{j} + \hat{k}$ **and is orthogonal to the vector** $-2\hat{i} + \hat{j} + \hat{k}$. **It is given that the projection of** $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$.

Sol.

A vector coplanar with $\vec{a}\ \&\ \vec{b}$ and orthogonal to $\vec{c}$ is parallel to the triple product,

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Hence $\vec{v} = \alpha[(-3(\hat{i} - 2\hat{j} + \hat{k}) + 3(\hat{i} + \hat{j} - 2\hat{k})] = 9\alpha(\hat{j} - \hat{k})$

Projection of $\vec{v}$ along $\hat{i} - \hat{j} + \hat{k} = \frac{\vec{v}.(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|} = 6\sqrt{3}$

$9\alpha(\hat{j} - \hat{k}).(\hat{i} - \hat{j} + \hat{k}) = 189\alpha(-1, -1) = 18$

$\Rightarrow \alpha = -1$ Ans. : $9(-\hat{j} + \hat{k})$

Ex.25 ABCD is a tetrahedron with A(–5, 22, 5); B(1, 2, 3); C(4, 3, 2); D(–1, 2, –3). Find $\vec{AB} \times (\vec{BC} \times \vec{BD})$. What can you say about the values of $(\vec{AB} \times \vec{BC}) \times \vec{BD}$ and $(\vec{AB} \times \vec{BD}) \times \vec{BC}$. Calculate the volume of the tetrahedron ABCD and the vector area of the triangle AEF where the quadrilateral ABDE and quadrilateral ABCF are parallelograms.

Sol.

$\vec{AB} \times (\vec{BC} \times \vec{BD}) = 0$; $(\vec{AB} \times \vec{BC}) \times \vec{BD} = 0$; $(\vec{AB} \times \vec{BD}) \times \vec{BC} = 0$;

Note that $\vec{AB}$; $\vec{BC}$; $\vec{BD}$ are mutually perpendicular.

Volume $= \frac{1}{6}[\vec{AB}, \vec{BC}, \vec{BD}] = \frac{220}{3}$ cu. units

Vector area of triangle AEF $= \frac{1}{2}\vec{AF} \times \vec{AE} = \frac{1}{2}\vec{BC} \times \vec{BD} = -3\hat{i} + 10\hat{j} + \hat{k}$

Product of Four Vectors

(a) Scalar Product of Four Vectors: The products already considered are usually sufficient for practical applications. But we occasionally meet with products of four vectors of the following types. Consider the scalar product of $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$. This is a number easily expressible in terms of the scalar products of the individual vectors. For, in virtue of the fact that in a scalar triple product the dot and cross may be interchanged, we may write

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{a} \cdot \vec{b} \times (\vec{c} \times \vec{d}) = \vec{a} \cdot ((\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

Writing this result in the form of a determinant,

we have $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

(b) Vector Product of Four Vectors:

Consider next the vector product of $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$. This is a vector at right angles to $\vec{a} \times \vec{b}$ and therefore coplanar with $\vec{a}$ and $\vec{b}$. Similarly it is coplanar with $\vec{a} \times \vec{b}$ It must therefore be parallel to the line of intersection of a plane parallel to $\vec{a}$ and $\vec{b}$ with another parallel to $\vec{c}$ and $\vec{d}$.

To express the product in $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{a}$ and $\vec{b}$ regard it as the vector triple product of $\vec{a}$ and $\vec{b}$ and $\vec{a}, \vec{b}$ and $\vec{m}$, where $\vec{m} = \vec{c} \times \vec{d}$. Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{m} = (\vec{a} \cdot \vec{m})\vec{b} - (\vec{b} \cdot \vec{m})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} \quad ......(1)$$

Similarly, regarding it as the vector product of $\vec{n}, \vec{c}$ and $\vec{d}$, where $\vec{n} = \vec{a} \times \vec{b}$ we may write it

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{n} \times (\vec{c} \times \vec{d}) = (\vec{n} \cdot \vec{d})\vec{c} - (\vec{n} \cdot \vec{c})\vec{d} = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} \quad .....(2)$$

Equating these two expressions we have a relation between the four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ viz.

$$[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} - [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = 0 \quad ...(3)$$

Ex.26 Show that
, $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ and deduce that $[\vec{a} \times \vec{b}\ ,\ \vec{b} \times \vec{c}\ ,\ \vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$.

Sol.

L.H.S. : $(\vec{b} \times \vec{c}) \times \vec{u} = (\vec{b} . \vec{u})\,\vec{c} - (\vec{c} . \vec{u})\,\vec{b} = [\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - 0$ $\qquad (\vec{u} = \vec{c} \times \vec{a})$

Hence $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ $\qquad$ taking dot with $\vec{a} \times \vec{b}$, $\qquad [\vec{a} \times \vec{b}\ ,\ \vec{b} \times \vec{c}\ ,\ \vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$.

Ex.27 Show that $\vec{a} \times ((\vec{q} \times \vec{c}) \times (\vec{p} \times \vec{b})) = \vec{b} \times ((\vec{p} \times \vec{c}) \times (\vec{q} \times \vec{a})) + \vec{c} \times ((\vec{p} \times \vec{a}) \times (\vec{q} \times \vec{b}))$

So

l.

consider $\vec{a} \times \left[(\vec{q} \times \vec{c}) \times (\vec{p} \times \vec{b})\right] = \vec{a} \times \left[(\vec{u} . \vec{b})\vec{p} - (\vec{u} . \vec{p})\vec{b}\right] = (\vec{a} \times \vec{p}).[\vec{q}\,\vec{c}\,\vec{b}] - (\vec{a} \times \vec{b}).[\vec{q}\,\vec{c}\,\vec{b}]$ $\qquad ......(1)$

similarly $\vec{b} \times \left[(\vec{p} \times \vec{c}) \times (\vec{q} \times \vec{a})\right] = (\vec{b} \times \vec{q}).[\vec{p}\,\vec{c}\,\vec{a}] - (\vec{b} \times \vec{a}).[\vec{p}\,\vec{c}\,\vec{q}]$ $\qquad ....(2)$

and $\vec{c} \times \left[(\vec{p} \times \vec{a}) \times (\vec{q} \times \vec{b})\right] = \vec{c} \times \left[\vec{u} \times \vec{v}\right] = (\vec{c} . \vec{v})\,\vec{u} - (\vec{c} . \vec{u})\,\vec{v} = [\vec{c}\,\vec{q}\,\vec{b}]\,(\vec{p} \times \vec{a}) - [\vec{c}\,\vec{p}\,\vec{a}] \times (\vec{q} \times \vec{b})$ $\qquad ....(3)$

Now $\quad (1) - (2) - (3) = 0 \Rightarrow$ result

K. VECTOR EQUATIONS

Ex.28 Solve the equation $\vec{x} \times \vec{a} = \vec{b}, (\vec{a} \cdot \vec{b} = 0)$.

Sol. From the vector product of each member with a, and obtain $\vec{a}^2\vec{x} - (\vec{a} \cdot \vec{x})\,\vec{a} = \vec{a} \times \vec{b}$.
The general solution, with λ as parameter, is $\vec{x} = \lambda\vec{a} + \vec{a} \times \vec{b}/\vec{a}^2$.

Ex.29 Solve the simultaneous equations $p\vec{x} + q\vec{y} = \vec{a},\ \vec{x} \times \vec{y} = \vec{b}, (\vec{a} \cdot \vec{b} = 0)$.

Sol. Multiply the first vectorially by $\vec{x}$, and substitute for $\vec{x} \times \vec{y}$ from the second. Then $q\vec{b} = \vec{x} \times \vec{a}$, which is of the same form as the equation in the preceding example.

Thus $\vec{x} = \lambda\vec{a} + q\vec{a} \times \vec{b}/\vec{a}^2$.
Substitution of this value in the first equation gives $\vec{y}$.

Ex.30 Solve $\vec{x} + \vec{a} + (\vec{x} \cdot \vec{b})\vec{c} = \vec{d}$

Sol. Multiply scalarly by $\vec{a}$. Then $(\vec{x} \cdot \vec{b})(\vec{a} \cdot \vec{c}) = \vec{a} \cdot \vec{d}$.

Substitute for $\vec{x} \cdot \vec{b}$ in (i) and obtain

$\vec{x} \times \vec{a} = \vec{d} - (\vec{a} \cdot \vec{d})\vec{c}/\vec{a} \cdot \vec{c}$

$= \vec{a} \times (\vec{d} \times \vec{c})/\vec{a} \cdot \vec{c}$.

$\vec{x} = \lambda\vec{a} + \vec{a} \times (\vec{a} \times (\vec{d} \times \vec{c}))/(\vec{a} \cdot \vec{c})\vec{a}^2$.

Ex.31 If

If $\vec{A}+\vec{B}=\vec{a}, \vec{A}.\vec{a}=1$ and $\vec{A}\times\vec{B}=\vec{b}$ then prove that

$$\vec{A} = \frac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2} \text{ and } \vec{B} = \frac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$$

Sol.

$\vec{A}+\vec{B}=\vec{a}$

taking dot with $\vec{a}$ $\vec{a}.\vec{B} = |\vec{a}|^2 - 1$ ...(1)

$\vec{A} \times \vec{B} = \vec{b}$

taking cross with $\vec{a}$ $(\vec{a}.\vec{B})\vec{A} - (\vec{a}.\vec{A})\vec{B} = \vec{a} \times \vec{b}$ $(|\vec{a}|^2 - 1)\vec{A} - \vec{B} = \vec{a} \times \vec{b}$ ....(2)

Solving (2) and $\vec{A} + \vec{B} = \vec{a}$ simultaneously we get the desired result.

Ex.32 Solve the vector equation in $\vec{x}$: $\vec{x} + \vec{x} \times \vec{a} = \vec{b}$.

Sol.

Taking dot with a = $\vec{x}.\vec{a} = \vec{b}.\vec{a}$ ...(1)

Taking cross with a = $\vec{x} \times \vec{a} + (\vec{x} \times \vec{a}) \times \vec{a} = \vec{b} \times \vec{a}$ ...(2)

$\vec{b} - \vec{x} + (\vec{x}.\vec{a})\vec{a} - (\vec{a}.\vec{a})\vec{z} = \vec{b} \times \vec{a}$ $(\vec{b} + \vec{a}(\vec{b}.\vec{a}) - \vec{b} \times \vec{a} = \vec{x}(1 + \vec{a}.\vec{a})$

$\Rightarrow \quad \vec{x} = \frac{1}{1 + \vec{a}.\vec{a}} \{\vec{b} + (\vec{a}.\vec{b})\vec{a} + \vec{a} \times \vec{b}\}$

Ex.33 Express a vector $\vec{R}$ as a linear combination of a vector $\vec{A}$ and another perpendicular to A and coplanar with $\vec{R}$ and $\vec{A}$.

Sol.

$\vec{A} \times (\vec{A} \times \vec{R})$ is a vector perpendicular to $\vec{A}$ and coplanar with $\vec{A}$ and $\vec{R}$.

Hence let,

$\vec{R} = \lambda\vec{A} + \mu\vec{A} \times (\vec{A} \times \vec{R})$ ...(1)

taking dot with $\vec{A}$, $\vec{R}.\vec{A} = \lambda\vec{A}.\vec{A}$

$\Rightarrow \quad \lambda = \frac{\vec{R}.\vec{A}}{\vec{A}.\vec{A}}$

again taking cross with $\vec{A}$

$$\vec{R} \times \vec{A} = \mu[\vec{A} \times (\vec{A} \times \vec{R})] \times \vec{A}$$

$$= \mu[(\vec{A} \cdot \vec{R})\vec{A} - (\vec{A} \cdot \vec{R})\vec{R}] \times \vec{A}$$

$$= -\mu(\vec{A} \cdot \vec{A})(\vec{R} \cdot \vec{A}) \quad \therefore \quad \mu = -\frac{1}{\vec{A} \cdot \vec{A}}$$

$$\text{Hence } \vec{R} = \left(\frac{\vec{R} \cdot \vec{A}}{\vec{A} \cdot \vec{A}}\right)\vec{A} - \frac{1}{\vec{A} \cdot \vec{A}} \vec{A} \times (\vec{A} \times \vec{R})]$$

Vector- Formulas

1. Definitions: A Vector may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. . A is called the initial point & B is called the terminal point. The magnitude of vector $|\vec{AB}|$ is expressed by $|\vec{AB}|$.

Zero vector a vector of zero magnitude i.e.which has the same initial & terminal point, is called a **Zero vector**. It is denoted by O.

Unit vector a vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by a^ symbolically $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.

Equal vectors two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

Collinear vectors two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors.** If they have the same direction they are named as like vectors otherwise.

unlike vectors. Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if,

$\vec{a}=K\vec{b}$, where K ∈ R Coplanar vectors a given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that "two vectors are always coplanar". position vector let O be a fixed origin, then the position vector of a point P is the vector →OP . If a & b & position vectors of two point A and B, then , →AB = b a

$- =$ pv of B $-$ pv of A .

2. Vector addition:

If two vectors $\vec{a} \& \vec{b}$ are represented by $\rightarrow \vec{OA} \& \vec{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\vec{OC}$, where OC is the diagonal of the parallelogram OACB.

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative)

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

$\vec{a}+0=\vec{a}=0+\vec{a}$

$\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

3. Multiplication of vector by scalars :

If $\vec{a}$ is a vector & m is a scalar, then m$\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This multiplication is called Scalar multiplication. If $\vec{a} \& \vec{b}$ are vectors & m, n are scalars, then:

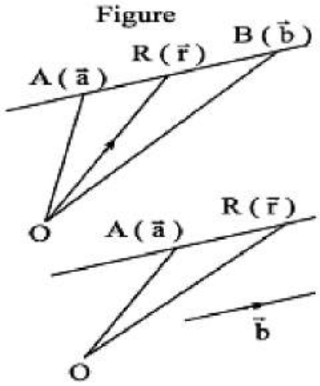
$m(a)=(a)m=ma$

$m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$

$m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

Figure



4. Section formula :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r}=\frac{n\vec{a}+m\vec{b}}{m+n}$.

note p.v..of mid point of AB = $\frac{\vec{a}+\vec{b}}{2}$

5. Direction Cosines

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ the angles which this vector makes with the +ve directions OX,OY & OZ are called Direction angles & their cosines are called the Direction cosines $\cos\alpha=\frac{a_1}{|\vec{a}|}$, $\cos\beta=\frac{a_2}{|\vec{a}|}$ $\cos\Gamma=\frac{a_3}{|\vec{a}|}$ Note that, $\cos^2$ αααα + $\cos^2$ ββββ + $\cos^2$ ΓΓΓΓ = 1.

6. Vector equation of a line:

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector then $\vec{b}$ its equation is, $\vec{r}=\vec{a}+t\mathrm{b}$

Note that the equations of the bisectors of the angles between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is: $\vec{r}=\vec{a}+t\left(\hat{b}+\hat{c}\right)$ & $\vec{r}=\vec{a}+p\left(\hat{c}-\hat{b}\right)$.

7. Test of collinearity :

Three points A,B,C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if & only if there exist scalars x , y , z not all zero simultaneously such that ; $x\vec{a}+y\vec{b}+z\vec{c}=0$, where x + y + z = 0.

8. Scalar product of two vectors:

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)$, note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

$\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2, \vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0 \ \ \vec{b} \ne 0)$

$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1;\ \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$.

Note: That vector component of $\vec{a}$ along $\vec{b} = \left(\frac{\vec{a}\cdot \vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$. the angle φ between $\vec{a}$ & $\vec{b}$ is given by $\cos\varphi = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \le \varphi \le \pi$.

if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Note : (i) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

(ii) Minimum values of $\vec{a}.\vec{b} = \vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$.

(iv) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vector $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} - \hat{b})$, $\lambda \in R^+$.

9. Vector product of two vectors :

(i) If $\vec{a} \& \vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a} \& \vec{b}$ such that $\vec{a}, \vec{b} \& \vec{n}$ forms a right handed screw system.

(ii) Lagranges Identity : for any two vectors $\vec{a} \& \vec{b}; (\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

(iii) Formulation of vector product in terms of scalar product:

The vector product $\vec{a} \times \vec{b}$ is the vector $\vec{c}$, such that.

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2}$ (ii) $\vec{c}\cdot\vec{a} = 0$; $\vec{c}\cdot\vec{b} = 0$ and (iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system.

(iv) $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a} \& \vec{b}$ are parallel (collinear) $(\vec{a} \neq 0, \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar..

$\vec{a}\times b \neq b\times\vec{a}$ (not commutative)

- ☞ $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$ where m is a scalar.
- ☞ $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)
- ☞ $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
- ☞ $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(v) ☞ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(vi) Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a} \& \vec{b}$.

(vii) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

- A vector of magnitude 'r ' & perpendicular to the palne of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(viii) Vector area If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC $\frac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$ The points A, B & C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$

Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$

10. Shortest distance between two lines: If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called SKEW LINES. For Skew lines the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$ i. e

$$\vec{LM} = |\text{Projection of } \vec{AB} \text{ on } \vec{LM}| = |\text{Projection of } \vec{AB} \text{ on } \vec{p}\times\vec{q}| = \left|\frac{\vec{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|.$$

1. The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e.

$(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$. $\Rightarrow [(\vec{b}-\vec{a})\ \vec{p}\ \vec{q}] = 0$

2. If two lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K\vec{b}$ i.e. they are parallel then $d = \left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

11. Scalar triple product / box product / mixed product :

The scalar triple product of three vectors $\vec{a}, \vec{b} \& \vec{c}$ is defined as :

$\vec{a}\times\vec{b}.\vec{c} = |\vec{a}||\vec{b}||\vec{c}|$ sin θ cos φ where θ is the angle between $\vec{a}\times\vec{b} \& \vec{c}$. it is also defined as $[\vec{a}\vec{b}\vec{c}]$, spelled as box product .Scalar triple product geometrically represents the volume of the parallelopiped whose three couterminous edges are represented by $\vec{a}, \vec{b} \& \vec{c}$ i.e. $V = [\vec{a}\vec{b}\vec{c}]$

In a scalar triple product the position of dot & cross can be interchanged i.e.
$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\vec{c}\vec{a}] = [\vec{c}\vec{a}\vec{b}]$

$\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then .

In general , if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\vec{m}\vec{n}]$ where $\vec{\ell}, \vec{m} \& \vec{n}$ are non coplanar vectors .

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\vec{b}\vec{c}] = 0$,

Note : If $\vec{a}.\vec{b}.\vec{c}$ are non –

coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system .

$[i\,j\,k] = 1$ ☞ $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ ☞ $[(\vec{a} + \vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

The volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b} \& \vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

The positon vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}, \vec{b}, \vec{c} \& \vec{d}$ are given by $\frac{1}{4}[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron .

12. Vector Triple Product : Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product .

Geometrical interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a} \& (\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a} \times (\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane $\vec{b} \& \vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector lies in the plane of $\vec{b} \& \vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b} \& \vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars .

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ ☞ $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

13. Linear combinations / Linearly Independence and Dependence of Vectors :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots$

is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$ for any x, y, z ...... ∈ R. We have the following results :

(a) **Fundamental theorem in plane :** Let $\vec{a},\vec{b}$ be non zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a},\vec{b}$ i.e. There exist some unique x,y ∈ R such that $x\vec{a}+y\vec{b}=\vec{r}$

(b) **Fundamental theorem in space :** Let $\vec{a},\vec{b},\vec{c}$ be non−zero, non−coplanar vectors in space. Then any vector $\vec{r}$, can be uniqìly expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i

There exist some unique x,y ∈ R such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.

(c) If $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are n non zero vectors, & $k_1, k_2, ...k_n$ are n scalars & if the linear combination

$k_1\vec{x}_1+k_2\vec{x}_2+........k_n\vec{x}_n=0 \Rightarrow k_1=0, k_2=0.....k_n=0$ then we say that vectors $\vec{x}_1,\vec{x}_2,......\vec{x}_n$

Linearly independent vectors

(d) If $\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are not Linearly independent then they are said to be Linearly dependent vectors $k_1\vec{x}_1+k_2\vec{x}_2+........+k_n\vec{x}_n=0$ & if there exists at least one kr ≠ 0 then

$\vec{x}_1,\vec{x}_2,......\vec{x}_n$ are said to be linearly dependent.

Note:

If $\vec{a}=3i+2j+5k$ then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system. $\hat{i}, \hat{j}, \hat{k}$ are Linearly independent set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=0 \Rightarrow K_1=0=K_2=K_3$.

Two vectors $\vec{a} \& \vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a} \& \vec{b}$ Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a} \& \vec{b}$ are linearly independent .

If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}, \vec{b}, \vec{c}] = 0$,

conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent.

14. Coplanarity of vectors:

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, $x + y + z + w = 0$.

15. Reciprocal system of vectors:

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

then the two systems are called Reciprocal System of vectors.

Note: $a' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $b' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $c' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

16. Equation of a plane

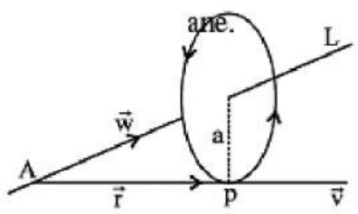
(a) The equation $(\vec{r} - \vec{r}_0).\vec{n} = 0$ represents a plane containing the point with p.v. $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane . $\vec{r}.\vec{n} = d$ is the general equation of a plane.

(b) Angle between the 2 planes is the angle between 2 normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

17. Application of vectors:

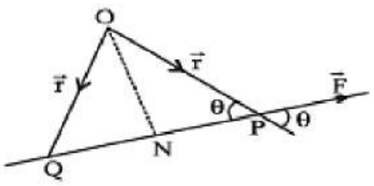
(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as

$\vec{W} = \vec{F}.\vec{s}$



(b) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V} = \vec{w} \times \vec{r}$ where $\vec{r}$ is the pv of the point P.

(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F} \& \vec{M}$ form a right handed system.



(d) Moment of the couple = $(\vec{r}_1 - \vec{r}_2) \times \vec{F}$ where $\vec{r}_1 \& \vec{r}_2$ are pv's of the point of the application of the forces $\vec{F} \& -\vec{F}$.